Studying entanglement-assisted entanglement transformation

2004 J. Phys. A: Math. Gen. 37 11439

(http://iopscience.iop.org/0305-4470/37/47/012)

J. Phys. A: Math. Gen. **37** (2004) 11439–11446 PII: S0305-4470(04)82500-9

Studying entanglement-assisted entanglement transformation

Li-Yi Hsu

Physics Division, National Center of Theoretical Sciences, Hsinchu, 30055, Taiwan, Republic of China
and
Department of Physics, Chung Yuan Christian University, Chungli, 32023, Taiwan, Republic of China

Received 18 June 2004, in final form 16 September 2004
Published 10 November 2004
Online at stacks.iop.org/JPhysA/37/11439
doi:10.1088/0305-4470/37/47/012

Abstract

In this paper, we study catalysis of entanglement transformations for n-level pure entangled states. We propose an algorithm of finding the required catalystic entanglement. We introduce several examples by way of demonstration. We evaluate the lower and upper bound of the required inequalities for deciding whether there are m-level appropriate catalyst states for entanglement transformations for two n-level pure entangled states.

PACS number: 03.67.Hk

Entanglement is the most distinguishing feature of quantum mechanics from classical mechanics. As pointed out by Schrödinger, the whole of a pure entangled state is a definite state, but the parts taken individually are not [1]. In the last decade, quantum information has been extensively explored. Studies of quantum entanglement have received considerable attention in recent years, with various applications in quantum cryptography [2], teleportation [3] and superdense coding [4]. It is well known that, using entanglement, we can accomplish many tasks that are impossible in classical information. Researchers now treat entanglement as a physical resource in quantum information. In addition, entanglement can be measured, mixed, distilled, diluted and transformed [5–8]. In this paper, we study the entanglement transformation between bipartite pure entangled states. To begin with, suppose that distant Alice and Bob share a bipartite system $|\psi_1\rangle \in \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ that comprises qubits A and B. As required in the following discussion, the dimensions of the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ are mn. These special state vectors $|\psi_1\rangle, |\psi_2\rangle, |\alpha\rangle$ and $|\beta\rangle$ can be spanned in the respective Hilbert subspaces. According to Schmidt decomposition, there exists an orthonormal basis $\{|i\rangle_A\}$ in $\mathcal{H}_A$ and $\{|i\rangle_B\}$ in $\mathcal{H}_B$, such that

$$|\psi_1\rangle = \sum_{i=1}^{n} \sqrt{x_i}|i\rangle_A \otimes |i\rangle_B, \tag{1}$$

where $x_1, \ldots, x_n$ are Schmidt coefficients such that $x_1 \geqslant \cdots \geqslant x_n \geqslant 0 = x_j, n < j \leqslant mn$, and $\sum_{i=1}^{n} x_i = 1$ [9]. In the following discussion, any bipartite system is expressed in the Schmidt decomposed form. Now Alice and Bob want to convert the entangled state $|\psi_1\rangle$ into the entangled state $|\psi_2\rangle$ with certainty under local operations and classical communication (LOCC), where

$$|\psi_2\rangle = \sum_{i=1}^{n} \sqrt{y_i} |i\rangle_A \otimes |i\rangle_B, \tag{2}$$

and the ordered Schmidt coefficients $y_1 \geqslant \cdots \geqslant y_n \geqslant 0 = y_j, n < j \leqslant mn$, and $\sum_{i=1}^{n} y_i = 1$. The necessary and sufficient conditions for the local transformation of pure bipartite entangled states with certainty under LOCC were presented by Nielsen [11].

Nielsen's theorem. A transformation $\mathbf{T}$ that converts $|\psi_1\rangle$ to $|\psi_2\rangle$ with certainty can be realized using LOCC iff $\{x_i\}$ can be majorized by $\{y_i\}$, that is, iff for all p, $1 \leqslant p \leqslant n$,

$$\sum_{i=1}^{p} x_i \leqslant \sum_{i=1}^{p} y_i. \tag{3}$$

We denote $|\psi_1\rangle \Rightarrow |\psi_2\rangle$ if we can convert $|\psi_1\rangle$ to $|\psi_2\rangle$ with certainty under LOCC. According to Nielsen's theorem, it is likely that neither of $|\psi_1\rangle$ and $|\psi_2\rangle$ can convert to each other with certainty under LOCC, which is denoted by $|\psi_1\rangle \not\Leftrightarrow |\psi_2\rangle$. Jonathan and Plenio [8] considered *entanglement-assisted local transformation* as follows. Suppose that Alice and Bob can be temporarily supplied with another catalytic entangled state $|\alpha\rangle \in \mathcal{H}' = \mathcal{H}'_A \otimes \mathcal{H}'_B$ that comprises qubits A' and B'. In general, $|\alpha\rangle = \sum_{i=1}^{m} \sqrt{\alpha_i} |i\rangle_{A'} \otimes |i\rangle_{B'}$. Note that the dimension of each Hilbert space, $\mathcal{H}'_A$ and $\mathcal{H}'_B$, is also mn. In addition, $\alpha_1 \geqslant \cdots \geqslant \alpha_{nm} \geqslant 0 = \alpha_j$, $m < j \leqslant mn$, and $\sum_{i=1}^{m} \alpha_i = 1$. Then, as Nielsen's theorem implies, they can perform

$$|\psi_1\rangle \otimes |\alpha\rangle \Rightarrow |\psi_2\rangle \otimes |\alpha\rangle \tag{4}$$

with certainty using LOCC [8]. Alice and Bob make use of the intermediate entanglement $|\alpha\rangle$ without destroying or altering it. Here the entangled state $|\alpha\rangle$ can be regarded as playing the catalyst role in a chemical reaction. Here we describe how to perform an entanglement-assisted local transformation based on Nielsen's theorem. Note that Alice and Bob possess the qubit pairs (A, A') and (B, B'), respectively. At first, Alice and Bob each perform the conditional operation O_1,

$$\begin{aligned}
&|s\rangle_{\text{source}}|t\rangle_{\text{target}} \to |s\rangle|(s-1)m+t\rangle && \text{if} \quad s \leqslant n \quad \text{and} \quad t \leqslant m(n-s+1),\\
&|s\rangle_{\text{source}}|t\rangle_{\text{target}} \to |s\rangle|(s-1)m+t-mn)\rangle && \text{if} \quad s \leqslant n \quad \text{and} \quad m(n-s+1) < t \leqslant mn\\
&|s\rangle_{\text{source}}|t\rangle_{\text{target}} \to |s\rangle|t\rangle \quad \text{otherwise},
\end{aligned} \tag{5}$$

where the source and target qubit pairs are (A, B) and (A', B'), respectively. Then Alice and Bob each perform the conditional operation O_2,

$$\begin{aligned}
&|s\rangle_{\text{source}}|t\rangle_{\text{target}} \to |s\rangle \left|t+s-\left\lceil\frac{s}{m}\right\rceil\right\rangle && \text{if} \quad t \leqslant mn-s+\left\lceil\frac{s}{m}\right\rceil,\\
&|s\rangle_{\text{source}}|t\rangle_{\text{target}} \to |s\rangle \left|t+s-\left\lceil\frac{s}{m}\right\rceil-mn\right\rangle && \text{if} \quad mn-s+\left\lceil\frac{s}{m}\right\rceil < t \leqslant mn,
\end{aligned} \tag{6}$$

where the source and target qubit pairs are (A', B') and (A, B), respectively. As a result, $|\psi_r\rangle \otimes |\alpha\rangle$ can become

$$\sum_{i=1}^{mn} \sqrt{w_{f(i)}} \sqrt{\alpha_{g(i)}} |\bar{i}\rangle_{\text{Alice}} \otimes |\bar{i}\rangle_{\text{Bob}}, \tag{7}$$

where $|\bar{i}\rangle_{\text{Alice}} = |ii\rangle_{AA'}$, $|\bar{i}\rangle_{\text{Bob}} = |ii\rangle_{BB'}$, and (r, w) can be $(1, x)$ or $(2, y)$, respectively. In addition,

$$f(i) = \frac{i - (i \bmod m) - m\delta_{0,i \bmod m}}{m} + 1, \tag{8}$$

$$g(i) = i \bmod m + m\delta_{0,i} \bmod_m, \tag{9}$$

where $\delta_{u,v}$ is equal to 1 if $u = v$ and 0 otherwise. As a result, according to Nielsen's theorem, we can identify whether $|\psi_1\rangle \otimes |\alpha\rangle$ can be converted into $|\psi_2\rangle \otimes |\alpha\rangle$ with certainty using LOCC. We call such transformation in equation (4) standard entanglement catalyzed transformation or standard entanglement catalysis. In addition, Feng *et al* considered mutual catalysis of entanglement transformations for pure entangled states. In other words, they considered the following entanglement-assisted local transformation,

$$|\psi_1\rangle \otimes |\alpha\rangle \Rightarrow |\psi_2\rangle \otimes |\beta\rangle, \tag{10}$$

where $|\beta\rangle$ is a particular entangled state and $|\alpha\rangle \nRightarrow |\beta\rangle$. Morikoshi explored the recovery of entanglement lost in entanglement manipulation [13].

In this paper, we focus on how to find the useful entanglement catalysts $|\alpha\rangle$ (and $|\beta\rangle$). As a starting point, we consider the ordered Schmidt coefficients of $|\psi_i\rangle \otimes |\alpha\rangle$ (and $|\psi_2\rangle \otimes |\beta\rangle$). (The tensor product notation '$\otimes$' is ignored in the rest of the paper.) We assume the Schmidt number of two-partite state $|\alpha\rangle \in \mathcal{H}' = \mathcal{H}'_A \otimes \mathcal{H}'_B$ to be 2. That is,

$$|\alpha\rangle = \sqrt{\alpha_1}|1\rangle|1\rangle + \sqrt{\alpha_2}|2\rangle|2\rangle, \tag{11}$$

where $\alpha_1 \geqslant \alpha_2 \geqslant 0$ and $\alpha_1 + \alpha_2 = 1$. For the simplest case, let n be 2. Obviously, there are two possible 'paths' for non-increasing ordered Schmidt coefficients of $|\psi_i\rangle|\alpha\rangle$: either

$$z_1\alpha_1 \geqslant z_1\alpha_2 \geqslant z_2\alpha_1 \geqslant z_2\alpha_2, \tag{12}$$

or

$$z_1\alpha_1 \geqslant z_2\alpha_1 \geqslant z_1\alpha_2 \geqslant z_2\alpha_2, \tag{13}$$

where z_i can be x_i or y_i. Here we define a path in which the non-increasing ordered Schmidt coefficients are arranged from left to right. The hyphen can be regarded as '$\geqslant$'. Note that equations (12) and (13) are two logically possible paths. Only one path is legal. To decide which path is legal as the real non-increasing ordered Schmidt coefficients of $|\psi_1\rangle|\alpha\rangle$, we have to compare $\frac{\alpha_2}{\alpha_1}$ and $\frac{z_2}{z_1}$. If $\frac{\alpha_2}{\alpha_1} \geqslant \frac{z_2}{z_1}$ $\left(\frac{\alpha_2}{\alpha_1} \leqslant \frac{z_2}{z_1}\right)$, the path in equation (12) ((13)) is legal. Figure 1 represents all possible paths in the $n = 4$ case. To find the respective legal paths in $|\psi_1\rangle|\alpha\rangle$ and $|\psi_2\rangle|\alpha\rangle$, we have to consider the order of all $\frac{x_j}{x_i}$, $\frac{y_j}{y_i}$ $(i < j)$ and $\frac{\alpha_2}{\alpha_1}$ values in the interval [0, 1]. In general, if $|\alpha\rangle$ is some entangled state with Schmidt number m, we have to consider the order of all $\frac{x_j}{x_i}$, $\frac{y_j}{y_i}$ and some necessary $\frac{\alpha_j}{\alpha i}$ $(i < j)$.

Now we consider how to find a useful catalyst entanglement. Here $|\alpha\rangle$ is an entangled state with Schmidt number 2. Equivalently, we can rephrase our problem as follows: With what ratio $\frac{\alpha_2}{\alpha_1}$ can we convert $|\psi_1\rangle \otimes |\alpha\rangle$ to $|\psi_2\rangle \otimes |\alpha\rangle$ with certainty under LOCC? First, we consider the value of $\frac{\alpha_2}{\alpha_1}$ in the interval [0, 1], which is divided into subintervals with possible endpoints 0, 1, $\frac{x_j}{x_i}$ and $\frac{y_j}{y_i}$ $(i < j)$. For any subinterval $[a, b]$, the endpoint a can be 0, $\frac{x_j}{x_i}$ or $\frac{y_j}{y_i}$ $(i < j)$, and the endpoint b can be 1, $\frac{x_j}{x_i}$ or $\frac{y_j}{y_i}$ $(i < j)$. As a consequence, neither $\frac{x_j}{x_i}$ nor $\frac{y_j}{y_i}$ $(i < j)$ can be in the open subinterval (a, b). Then we investigate whether there is a useful region of $\frac{\alpha_2}{\alpha_1}$, in a subinterval, to convert $|\psi_1\rangle \otimes |\alpha\rangle$ to $|\psi_2\rangle \otimes |\alpha\rangle$ with certainty under LOCC. The advantage is that, for a subinterval, the ordered Schmidt coefficients of $|\psi_1\rangle \otimes |\alpha\rangle$ and $|\psi_2\rangle \otimes |\alpha\rangle$ are exactly determined, respectively. Obviously, there are $n^2 - n + 1$ subintervals

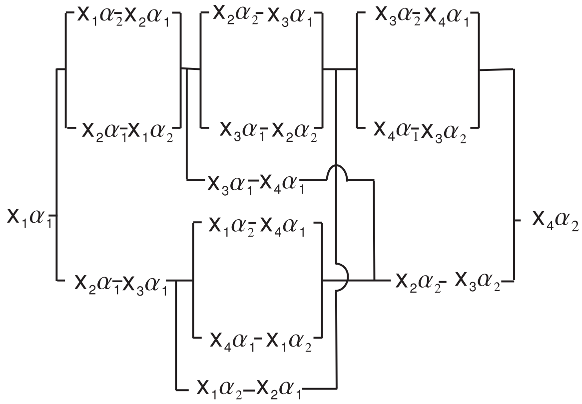

Figure 1. All logically possible paths of ordered Schmidt coefficients of $|\psi_1\rangle|\alpha\rangle$, where the Schmidt number of $|\alpha\rangle$ is 2 and that of $|\psi_1\rangle$ is 4.

if all $\frac{x_j}{x_i}$ and $\frac{y_j}{y_i}$ $(i < j)$ are different. Therefore, we have to consider $n^2 - n + 1$ cases. Here we assume that $|\psi_1\rangle \otimes |\alpha\rangle \Rightarrow |\psi_2\rangle \otimes |\alpha\rangle$, where $\frac{\alpha_2}{\alpha_1} \in \left[\frac{z_j}{z_i}, \frac{z_{j'}}{z_{i'}}\right]$, z and z' can be x or y and $i \leqslant j$ and $i' \leqslant j'$. Hence the order of the Schmidt coefficients of $|\psi_1\rangle \otimes |\alpha\rangle$ and $|\psi_2\rangle \otimes |\alpha\rangle$ is exactly determined. Then we consider the majorization conditions in equation (3). That is, we consider all inequalities similar in equation (3) in terms of $\frac{\alpha_2}{\alpha_1}$. As a result, we can derive many inequalities, such as $\frac{\alpha_2}{\alpha_1} \leqslant s_1, \ldots, \frac{\alpha_2}{\alpha_1} \leqslant s_a$ and $\frac{\alpha_2}{\alpha_1} \geqslant l_1, \ldots, \frac{\alpha_2}{\alpha_1} \geqslant l_b$, where $s_1, \ldots, s_a$ and $l_1, \ldots, l_b$ are functions of all x_i and y_i, which are already known. If

$$\max\left\{l_1, \ldots, l_b, \frac{z_j}{z_i}\right\} = B \leqslant A = \min\left\{s_1, \ldots, s_a, \frac{z_{j'}}{z_{i'}}\right\}, \tag{14}$$

we have a useful catalyst region $B \leqslant \frac{\alpha_2}{\alpha_1} \leqslant A$. In this way, we investigate all possible subintervals to find out all possible catalyst regions. We consider the following example.

Example 1. Suppose there are two entangled states

$$|\psi_1\rangle = \sqrt{0.4}|1\rangle|1\rangle + \sqrt{0.4}|2\rangle|2\rangle + \sqrt{0.1}|3\rangle|3\rangle + \sqrt{0.1}|4\rangle|4\rangle \tag{15}$$

and

$$|\psi_2\rangle = \sqrt{0.5}|1\rangle|1\rangle + \sqrt{0.25}|2\rangle|2\rangle + \sqrt{0.25}|3\rangle|3\rangle. \tag{16}$$

Find the appropriate catalyst state $|\alpha\rangle$ [8].

(x_1, x_2, x_3, x_4) and (y_1, y_2, y_3, y_4) are (0.4, 0.4, 0.1, 0.1) and (0.5, 0.25, 0.25, 0), respectively. Obviously, we have

$$x_1 \leqslant y_1 \tag{17}$$

and

$$x_1 + x_2 \geqslant y_1 + y_2. \tag{18}$$

As a result, $|\psi_1\rangle \nRightarrow |\psi_2\rangle$. Now we consider the appropriate catalyst entanglement. Before exploring all inequalities, we can make some observations. For example, the second largest Schmidt coefficients of $|\psi_1\rangle|\alpha\rangle$ and $|\psi_2\rangle|\alpha\rangle$ cannot be $x_2\alpha_1$ and $y_2\alpha_1$ simultaneously. Otherwise, $|\psi_1\rangle \Rightarrow |\psi_2\rangle$ cannot be achieved due to equation (17). Similarly, it is easy to verify that the fourth largest Schmidt coefficients of $|\psi_1\rangle|\alpha\rangle$ and $|\psi_2\rangle|\alpha\rangle$ cannot be $x_2\alpha_2$ and $y_2\alpha_2$ simultaneously due to equation (18). Therefore, some possible paths in figure 1 are excluded.

In addition, as previously mentioned, we divide the interval [0, 1] into subintervals, **1**, **2**, **3**, which are [0.5, 1], [0.25, 0.5] and [0, 0.25], respectively. If $\frac{\alpha_2}{\alpha_1}$ falls in either the subinterval **2** or **3**, the second largest Schmidt coefficients of $|\psi_1\rangle|\alpha\rangle$ and $|\psi_2\rangle|\alpha\rangle$ are $x_2\alpha_1$ and $y_2\alpha_1$, respectively, which violates equation (17). As a result $\frac{\alpha_2}{\alpha_1}$ can only be in subinterval **1**. That is, $0.5 \leqslant \frac{\alpha_2}{\alpha_1} \leqslant 1$. The ordered Schmidt coefficients of $|\psi_1\rangle|\alpha\rangle$ and $|\psi_2\rangle|\alpha\rangle$ are

$$x_1\alpha_1 \geqslant x_2\alpha_1 \geqslant x_1\alpha_2 \geqslant x_2\alpha_2 \geqslant x_3\alpha_1 \geqslant x_4\alpha_1 \geqslant x_3\alpha_2 \geqslant x_4\alpha_2 \tag{19}$$

and

$$y_1\alpha_1 \geqslant y_1\alpha_2 \geqslant y_2\alpha_1 \geqslant y_3\alpha_1 \geqslant y_2\alpha_2 \geqslant y_3\alpha_2\ (y_4 = 0),$$

respectively. With straight algebra on equation (3), we have

$$\frac{3}{5} \leqslant \frac{\alpha_2}{\alpha_1} \leqslant \frac{2}{3}. \tag{20}$$

In Jonathan and Plenio's paper, only the solution $\frac{\alpha_2}{\alpha_1} = \frac{2}{3}$ is considered. In our consideration, we can find all appropriate catalyst $\frac{\alpha_2}{\alpha_1}$ ratios. Moreover, we directly determine the corresponding ordered Schmidt coefficients.

Next we consider another example of mutual catalysis entanglement.

Example 2. Suppose there are two entangled states

$$|\psi_1\rangle = \sqrt{0.4}|1\rangle|1\rangle + \sqrt{0.36}|2\rangle|2\rangle + \sqrt{0.14}|3\rangle|3\rangle + \sqrt{0.1}|4\rangle|4\rangle \tag{21}$$

and

$$|\psi_2\rangle = \sqrt{0.5}|1\rangle|1\rangle + \sqrt{0.25}|2\rangle|2\rangle + \sqrt{0.25}|3\rangle|3\rangle. \tag{22}$$

Find the useful entanglement states $|\alpha\rangle$ and $|\beta\rangle$, such that $|\alpha\rangle \nRightarrow |\beta\rangle$ but $|\psi_1\rangle|\alpha\rangle \Rightarrow |\psi_2\rangle|\beta\rangle$ [12].

Here $|\alpha\rangle$ and $|\beta\rangle$ are entangled states with Schmidt number 2. That is, $|\alpha\rangle = \sqrt{\alpha_1}|55\rangle + \sqrt{\alpha_2}|66\rangle$ and $|\beta\rangle = \sqrt{\beta_1}|55\rangle + \sqrt{\beta_2}|66\rangle$. We assume that $\alpha_1 > \alpha_2$ and $\beta_1 > \beta_2$. Since $|\alpha\rangle \nrightarrow |\beta\rangle$, we have $\alpha_1 > \beta_1 > 0.5$. In addition, we can write down the necessary condition for $|\psi_1\rangle|\alpha\rangle \Rightarrow |\psi_2\rangle|\beta\rangle$

$$x_1\alpha_1 \leqslant y_1\beta_1. \tag{23}$$

That is, $0.8\alpha_1 \leqslant \beta_1$. As a result, we set

$$\beta_1 = p\alpha_1, \qquad 0.8 \leqslant p < 1. \tag{24}$$

Then we consider the majorization conditions, which lead to inequalities in terms of p and $\frac{\alpha_2}{\alpha_1}$. Similarly, we can divide the [0, 1] into eight subintervals. Suppose that the second largest Schmidt coefficients of $|\psi_1\rangle|\alpha\rangle$ and $|\psi_2\rangle|\alpha\rangle$ are $x_2\alpha_1$ and $y_2\beta_1$, respectively, The majorization condition requires that

$$x_1\alpha_1 + x_2\alpha_1 \leqslant y_1\beta_1 + y_2\beta_1. \tag{25}$$

As a result, we have

$$1 < \frac{x_1 + x_2}{y_1 + y_2} \leqslant p, \tag{26}$$

which contradicts equation (25). As a result, there are three possible conditions for $\left(\frac{\alpha_2}{\alpha_1}, \frac{\beta_2}{\beta_1}\right)$. With straightforward algebra, solutions for appropriate α_1 and p are

$$\frac{13}{25\alpha_1} \leqslant p < 1, \qquad \frac{13}{25} < \alpha_1 \leqslant \frac{23}{36}, \tag{27}$$

where the corresponding paths of $|\psi_1\rangle|\alpha\rangle$ can be

$$x_1\alpha_1 \geqslant x_1\alpha_2 \geqslant x_2\alpha_1 \geqslant x_2\alpha_2 \geqslant x_3\alpha_1 \geqslant x_3\alpha_2 \geqslant x_4\alpha_1 \geqslant x_4\alpha_2, \tag{28}$$

$$x_1\alpha_1 \geqslant x_2\alpha_1 \geqslant x_1\alpha_2 \geqslant x_2\alpha_2 \geqslant x_3\alpha_1 \geqslant x_3\alpha_2 \geqslant x_4\alpha_1 \geqslant x_4\alpha_2, \tag{29}$$

or

$$x_1\alpha_1 \geqslant x_2\alpha_1 \geqslant x_1\alpha_2 \geqslant x_2\alpha_2 \geqslant x_3\alpha_1 \geqslant x_4\alpha_1 \geqslant x_3\alpha_2 \geqslant x_4\alpha_2. \tag{30}$$

The other solution is

$$\frac{36}{25} - \frac{2}{5\alpha_1} \leqslant p < 1, \qquad \frac{23}{36} \leqslant \alpha_1 < \frac{50}{76}, \tag{31}$$

where the corresponding path of $|\psi_1\rangle|\alpha\rangle$ is the same as 30. For the above solutions, the corresponding path of $|\psi_2\rangle|\beta\rangle$ is

$$y_1\beta_1 \geqslant y_1\beta_2 \geqslant y_2\beta_1 \geqslant y_3\beta_1 \geqslant y_2\beta_2 \geqslant x_3\beta_2. \tag{32}$$

In the Feng *et al* paper, only the solution $(\alpha_1, \alpha_2, \beta_1, \beta_2) = (0.6, 0.4, 0.55, 0.45)$ is considered.

However, it is not always possible to perform mutual entanglement catalysis using entangled states as catalysts with Schmidt number 2. Feng *et al* considered $|\psi_1\rangle$ and $|\psi_2\rangle$ to be

$$\sqrt{0.33}|11\rangle + \sqrt{0.32}|22\rangle + \sqrt{0.3}|33\rangle + \sqrt{0.05}|44\rangle, \tag{33}$$

and

$$\sqrt{0.6}|11\rangle + \sqrt{0.2}|22\rangle + \sqrt{0.14}|33\rangle + \sqrt{0.06}|44\rangle, \tag{34}$$

respectively [12]. With lengthy algebra, we verify that there are no entangled states $|\alpha\rangle$ and $|\beta\rangle$ with Schmidt number 2 for mutual catalysis of entanglement transformation even if the necessary condition

$$\beta_1 \ln \beta_1 + \beta_2 \ln \beta_2 > \alpha_1 \ln \alpha_1 + \alpha_2 \ln \alpha_2, \tag{35}$$

is satisfied [12].

In general, we can determine whether there are entangled states $|\alpha\rangle$ (and $|\beta\rangle$) with Schmidt number m for entangled states $|\psi_1\rangle$ and $|\psi_2\rangle$ with Schmidt number n in a similar way. We describe our algorithm as follows. (1) As previously mentioned, we divide the interval [0, 1] into subintervals. As a result, none of $\frac{x_j}{x_i}$ and $\frac{y_j}{y_i}$ $(i < j)$ falls into any open subinterval. (2) Find the legal paths of $|\psi_1\rangle|\alpha\rangle$ and $|\psi_2\rangle|\alpha\rangle$ ($|\psi_2\rangle|\beta\rangle$), respectively. (3) Consider all inequalities for the majorization. Here we approximate the lower and upper bounds of the total inequalities to be considered. To search appropriate catalyst entangled states with Schmidt number m, at least we have to consider all independent ratios $\frac{\alpha_{i+1}}{\alpha_i}$ (and $\frac{\beta_{i+1}}{\beta_i}$), $i = 1, \ldots, m-1$. Each ratio can fall into any subinterval. Here we consider standard entanglement catalysis. Suppose that all ratios $\frac{x_j}{x_i}$ and $\frac{y_j}{y_i}$ $(i < j)$ are different and, therefore, there are $n(n-1)+1$ subintervals. As a result, there are at least $(n^2 - n + 1)^{m-1}$ possible cases for the $(\frac{\alpha_2}{\alpha_1}, \frac{\alpha_3}{\alpha_2}, \ldots, \frac{\alpha_n}{\alpha_{n-1}})$ distribution. For any $\frac{\alpha_{i+1}}{\alpha_i}$ in some subinterval, two inequalities should be considered. In addition, we have to consider $(nm-1)$ majorization inequalities. Therefore, for each case we have to consider $n(m+2) - 1$ inequalities at least. For large n and m, at least we have to consider about $n^{2m-1}m$ inequalities to decide whether there are appropriate catalyst states with Schmidt number m for standard catalysis of entanglement transformations for two pure entangled states. On the other hand, even the $\left(\frac{\alpha_2}{\alpha_1}, \frac{\alpha_3}{\alpha_2}, \ldots, \frac{\alpha_n}{\alpha_{n-1}}\right)$ distribution is known, it is not sufficient to decide the legal paths. In fact, there are $\frac{1}{2}m(m-1)\frac{\alpha_j}{\alpha_i}$ $(i < j)$ ratios and only $(m-1)$ of these ratios are independent. To approximate the upper bound, we assume that all $\frac{\alpha_j}{\alpha_i}$ are independent. As a

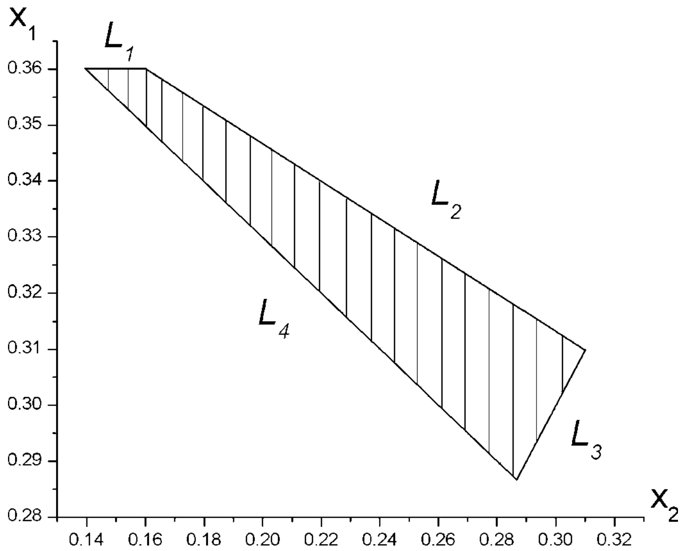

Figure 2. The slash area shows all possible (x_2, x_1) of $|\psi_1\rangle$ in equation (37) for the entanglement catalysis $|\psi_1\rangle|\psi_2\rangle \to |\psi_2\rangle|\psi_2\rangle$. Here $L_1 : x_1 = 0.36$, $L_2 : 3x_1 + x_1 = 1.24 L_3 : x_1 = x_2$ and $L_4 : 2x_1 + x_2 = 0.86$.

consequence, it is easy to verify that, at most, we need to consider about $n^{m^2-m+1}m$ inequalities to decide whether there are appropriate catalyst states with Schmidt number m. Similarly, we need at least (at most) $n^{4m-3}m$ $(n^{2m(m-1)+1}m)$ to decide whether there are appropriate catalyst states with Schmidt number m for mutual catalysis of entanglement transformations for two pure entangled states when n and m become large. For $m = 2$, we may need about n^3 inequalities to find appropriate catalyst states.

Finally, we consider the following example.

Example 3. Suppose

$$|\psi_2\rangle = \sqrt{0.48}|1\rangle|1\rangle + \sqrt{0.24}|2\rangle|2\rangle + \sqrt{0.14}|3\rangle|3\rangle + \sqrt{0.14}|4\rangle|4\rangle \tag{36}$$

$(y_1 = 0.48,\ y_2 = 0.24,\ y_3 = y_4 = 0.14)$. Find the useful entanglement states $|\psi_1\rangle$,

$$|\psi_1\rangle = \sqrt{x_1}|1\rangle|1\rangle + \sqrt{x_1}|2\rangle|2\rangle + \sqrt{x_2}|3\rangle|3\rangle + \sqrt{x_3}|4\rangle|4\rangle + \sqrt{x_3}|5\rangle|5\rangle \tag{37}$$

$\left(x_3 = \frac{1-2x_1-x_2}{2}\right)$, such that

$$|\psi_1\rangle \nRightarrow |\psi_2\rangle \tag{38}$$

but

$$|\psi_1\rangle|\psi_2\rangle \to |\psi_2\rangle|\psi_2\rangle. \tag{39}$$

[14].

Jensen and Schack introduced this example as quantum authentication and authenticated quantum key distribution [14]. Here we just focus on all solutions of $|\psi_1\rangle$ in equation (37). In this example, it is $|\psi_1\rangle$ rather than the catalyst entanglement that is unknown. According to equations (38) and (39), it is easy to verify that (i) $x_1 \leqslant \frac{y_1+y_2}{2}$ and (ii) $2x_1 + x_2 \geqslant y_1 + y_2 + y_3$. With straight algebra, we can find all possible x_1 and x_2, as shown in figure 2. In Jensen and Schack's paper, only the solution $(x_1, x_2, x_3) = (0.31, 0.30, 0.04)$ is considered.

In conclusion, we demonstrate how to find catalyst entanglement with Schmidt number 2 for standard and mutual catalyses of entanglement transformations for pure entangled states. We propose an algorithm to determine whether we can perform standard or mutual entanglement catalysis using catalyst entanglement with Schmidt number *m*. The lower and upper bounds of the required inequalities for solving such problems are evaluated. We conjecture that there are no efficient criteria for deciding whether there are appropriate catalyst states with Schmidt number *m* for standard or mutual catalyses of entanglement transformations with Schmidt number *n* for two pure entangled states when *n* and *m* become large.

References

[1] Schrödinger E 1935 *Naturwissenschaften* **48** 807
Schrödinger E 1935 *Naturwissenschaften* **49** 823, 844
Schrödinger E 1980 *Proc. Am. Phil. Soc.* **124** 323 (Engl. Transl.)
[2] Gisin N, Ribordy G, Tittel W and Zbinden H 2002 *Rev. Mod. Phys.* **74** 145
[3] Bennett C H, Brassard G, Crépeau C, Jozsa R, Peres A and Wootters W K 1993 *Phys. Rev. Lett.* **70** 1895
[4] Bennett C H and Wiesner S J 1992 *Phys. Rev. Lett.* **69** 2881
[5] Wei T C, Nemoto K, Goldbart P M, Kwiat P G, Munro W J and Verstraete F 2003 *Phys. Rev.* A **67** 022110
[6] Bennett C H, Bernstein H J, Popescu S and Schumacher B 1996 *Phys. Rev.* A **53** 2046
[7] Lo H K and Popescu S 2001 *Phys. Rev.* A **63** 022301
[8] Jonathan D and Plenio M B 1999 *Phys. Rev. Lett.* **83** 3566
[9] Ekert A and Knight P L 1995 *Am. J. Phys.* **63** 415
[10] Acín A, Andrianov A, Costa L, Jané E, Latorre J I and Tarrach R 2000 *Phys. Rev. Lett.* **85** 1560
[11] Nielsen M A 1999 *Phys. Rev. Lett.* **83** 436
[12] Feng X L, Wang Z Y and Xu Z Z 2002 *Phys. Rev.* A **65** 022307
[13] Morikoshi F 1999 *Phys. Rev. Lett.* **84** 3189
[14] Jensen J G and Schack R 2000 *Preprint* quant-ph/0003104